

Hypersurface-orthogonal generators of an orthogonally transitive transitive G_2I , topological identifications, and axially and cylindrically symmetric spacetimes

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Abstract

A criterion given by Castejon-Amenedo and MacCallum (1990) for the existence of (locally) hypersurface-orthogonal generators of an orthogonally-transitive two-parameter Abelian group of motions (a G_2I) in spacetime is re-expressed as a test for linear dependence with constant coefficients between the three components of the metric in the orbits in canonical coordinates. In general, it is shown that such a relation implies that the metric is locally diagonalizable in canonical coordinates, or has a null Killing vector, or can locally be written in a generalized form of the ‘windmill’ solutions characterized by McIntosh. If the orbits of the G_2I have cylindrical or toroidal topology and a periodic coordinate is used, these metric forms cannot in general be realized globally as they would conflict with the topological identification. The geometry then has additional essential parameters, which specify the topological identification. The physical significance of these parameters is shown by their appearance in global holonomy and by examples of exterior solutions where they have been related to characteristics of physical sources. These results lead to some remarks about the definition of cylindrical symmetry.

Keywords: cylindrical symmetry, stationary, Killing vectors, topology, holonomy

1 Introduction

This paper considers spacetime metrics with a pair of orthogonally-transitive commuting Killing vectors (KVs), generating a group of type G_2I (Abelian). The most often considered cases are those in which it is possible to write the metric locally in the form (see Kramer *et al* (1980), pages 194 and 220)

$$ds^2 = e^{-2U} [e^{2k} (d\rho^2 - dt^2) + \rho^2 d\phi^2] + e^{2U} (dz + A d\phi)^2 \quad (1.1)$$

where U , k and A are real functions of t and ρ only, if the G_2 acts on spacelike surfaces, or

$$ds^2 = e^{-2U} [e^{2k} (d\rho^2 + dz^2) + \rho^2 d\phi^2] - e^{2U} (dt + A d\phi)^2, \quad (1.2)$$

where U , k and A are real functions of z and ρ only, if the G_2 acts on timelike surfaces. (These forms, or the special cases considered by Hoffman (1969) which will not be discussed here, are valid for vacuum and some other energy-momentum tensors. In certain circumstances, such as the existence of regular axis points, the orthogonal transitivity of the G_2 can be proved rather than assumed, but I do not pursue that point here.) The two forms just given can both be written as

$$ds^2 = e^{-2U} [e^{2k} (d\rho^2 - \zeta dY^2) + \rho^2 d\phi^2] + \zeta e^{2U} (dX + A d\phi)^2 \quad (1.3)$$

where $\zeta = \pm 1$, U , k and A are functions of Y and ρ only, and, if $\zeta = 1$, then $Y = t$ and $X = z$, while if $\zeta = -1$, then $Y = z$, $X = t$. The KVs $\xi = \partial/\partial X$ and $\eta = \partial/\partial \phi$ form a basis of the generators of the G_2 .

One may also write the metric, in slightly more general coordinates which no longer restrict the energy-momentum, as

$$ds^2 = e^\mu dr^2 - \zeta e^\nu dY^2 + \ell d\phi^2 + 2m d\phi dX + \zeta f dX^2 \quad (1.4)$$

where the metric components depend only on r and Y . The form (1.3) is the case where $r = \rho$ and $\mu = \nu = U$; the metric coefficients of the cases above are included in this form, with corresponding values $\ell \equiv \chi = (\rho^2 + \zeta(Ae^{2U})^2)e^{-2U}$ (χ being the notation of Castejon-Amenedo and MacCallum (1990)), $m = \zeta Ae^{2U}$, and $f = e^{2U}$. One can of course choose new mutually orthogonal coordinates $\hat{r} = \hat{r}(r, Y)$, $\hat{Y} = \hat{Y}(r, Y)$ without altering the general form (1.4). I shall refer to equation (1.4) for brevity as the Lewis form, cf. Da Silva *et al* (1995b), since it was used by Lewis (1932) with $\zeta = -1$ (equation (1.1) there). I define ρ in all cases by $\rho^2 = f\ell - \zeta m^2$.

When, for example, investigating whether a gravitational wave solution is linearly polarized, it is of interest to know whether among the generators of the G_2 there is a hypersurface-orthogonal Killing vector¹. In Castejon-Amenedo and MacCallum (1990), it was shown that if neither ξ nor η are themselves hypersurface-orthogonal (HSO), then, among the generators of the G_2 , there is one which is HSO and non-null if and only if an equation

$$\frac{1}{C} = -\zeta \frac{B\chi + \zeta Ae^{2U}}{e^{2U}(1 + BA)} \quad (1.5)$$

holds, with distinct real constants B and C . In fact, if there is one, there must be a second, orthogonal to the first, and the two HSO KVs can be taken to be $\xi + B\eta$ and $\xi + C\eta$. Equation (1.5) can be stated in a simpler form as the requirement that the metric coefficients of the Lewis form obey a linear relation

$$\zeta f(r, Y) + am(r, Y) + b\ell(r, Y) = 0. \quad (1.6)$$

with constant coefficients $a = (B + C)$ and $b = BC$. Despite the simplicity of this form of the criterion, it did not appear in a general form in any earlier paper known to

¹If the G_2 under consideration is not the maximal group of motions, there may be an HSO KV which is not one of its generators. This happens, for example, in some stationary cylindrically symmetric metrics discussed later in this paper, but the possibility will not be discussed fully here.

me. (However, similar statements in the context of cylindrically symmetric stationary vacuum and electrovacuum solutions were given by Arbex and Som (1973) and Som and Santos (1978), and, for the case of spacelike orbits, a form of the general criterion has just appeared as equation (9) in Mars and Senovilla (1997): I thank J.M.M. Senovilla for drawing the latter to my attention.)

In this form it is often immediately apparent whether the criterion is satisfied or not, by inspecting whether or not the functions f , m and ℓ contain only two functions of r and Y linearly independent over \mathbb{R} . When the criterion is hard to test, due to the complexity of the functions being considered, one can find the coefficients in equation (1.6) by evaluation at specific r and Y : consistency of the equations at three (or more) such evaluation points is required, and of course the resulting B and C must be valid at all points. Clearly, if the criterion (1.6) is satisfied, B and C will simply be the roots of

$$w^2 - aw + b = 0. \quad (1.7)$$

This prompts the question: what if the functions f , m and ℓ are linearly dependent but the associated quadratic does not have distinct real roots? In Section 2 I show that if the functions f , m and ℓ are linearly dependent with constant coefficients (necessarily real), then either there is a pair of non-null HSO KVs (and the metric is locally diagonalizable²), or there is a null HSO KV, which one can regard as the limiting case where the two non-null HSO KVs coincide, or the metric can be expressed in a generalization of the McIntosh ‘windmill’ form for vacuum solutions (McIntosh, 1992). Note that this formulation avoids the difficulties of expressing, within some version of the above formulae, the cases where one or both Killing vectors have already been aligned with hypersurface-orthogonal ones. The last of the three cases can be viewed as having a pair of complex conjugate hypersurface-orthogonal KVs. For certain vacuum metrics, this result is implicit in section 18.4 of Kramer *et al* (1980) and references cited therein (see also Som and Santos (1978)) but again the general statement appears to be new.

These considerations are purely local. They will therefore be consistent with the results one can obtain from the local characterization of the metric in terms of Cartan scalars (Karlhede, 1980; MacCallum and Skea, 1994; Paiva *et al*, 1993; Da Silva *et al*, 1995b). However, the metric may have parameters which are important globally but do not appear in the Cartan scalars. This is considered further in section 3 for the case of axisymmetric metrics, where the coordinate suggestively named ϕ is normally assumed to be periodic. Choosing ϕ to have the usual period 2π , one finds that the locally equivalent geometries with distinct identifications in the G_2 orbits, and the choices of ignorable coordinates consistent with them, can be specified by three parameters. On careful consideration of possible coordinate choices it turns out that only two of the three parameters are essential, in that they cannot be removed by permissible changes of coordinates. (In this paper, ‘essential’ refers only to unique characterization of the geometry, not to any other purpose.)

The three parameters can be interpreted, in terms of the identification, as follows.

²Throughout this paper, ‘diagonalizable’ means ‘diagonalizable in a holonomic frame two of whose coordinates are ignorable’, i.e. in the form (1.4) with $m = 0$. Two-dimensional metrics, being conformally flat, can always be diagonalized, but not necessarily in a way consistent with the form (1.4). Also, metrics of any dimension can be diagonalized in non-holonomic tetrads, e.g. orthonormal tetrads. In older papers where this fact was used, confusion was sometimes created, at least for a modern reader, by the practice of notating the basis one-forms as if they were coordinate differentials, for example in equations (3.1) of Lewis (1932).

If the metric in the (ϕ, X) plane takes a standard form (as described in Section 2) in coordinates $(\hat{\phi}, \hat{X})$, two of the three parameters specify a point $(\hat{\phi}_0, \hat{X}_0)$ which is to be identified with the origin, and the third (the inessential one) specifies the direction of parallel lines through the origin and $(\hat{\phi}_0, \hat{X}_0)$ along which the identification is to take place. To understand the relation between locally equivalent but globally inequivalent axisymmetric metrics with the Lewis form, one can imagine unrolling the cylinder $0 \leq \phi \leq 2\pi$, $-\infty < X < \infty$ into a plane and then rolling it up again in a different way.

From a physical point of view one would wish to associate the additional essential parameters with curvature, and this is discussed in section 4. The parameters cannot change the values of the Cartan scalars defined by the Riemann tensor and its derivatives at a point, and this directs attention to the possible global holonomy found by taking suitable closed curves, i.e. the ‘gravitational Aharonov-Bohm effect’ of Marder (1959) and Stachel (1982). Stachel points out that for a given metric form, the linear holonomy will depend only on that metric’s curvature unless the region in which it is defined is not simply connected (has non-zero first Betti number).

Stachel does not give explicit general formulae for this holonomy (though he calculates some specific results in Stachel (1984)). These are derived in Section 4. In general the two essential parameters of the identification do appear in these formulae. The metrics studied may be matched to some regular source in an interior region with a different metric; in this case, the holonomy not due to the curvature of the exterior is due to the curvature in the region occupied by the source. If the solutions are valid for all points except some singular axis (a concept which as yet lacks precise definition, see Mars and Senovilla (1995)), the holonomy may be ascribed to singular sources on the axis. Vickers (1987) and Tod (1994) have considered the linear and affine holonomy of spacetimes constructed by identifications on flat space, and Tod has remarked that the non-zero affine holonomy could be considered to arise from a distributional torsion on the axis and can also be viewed as an example where a non-local effect of a symmetric connection mimics a local effect of a non-symmetric connection. Wilson and Clarke (1996) considered the general behaviour of holonomy at an axis.

When there is a definite interior matched to the region being considered, all the essential parameters will be fixed by the matching, using the coordinate-free Darmois form of the boundary conditions. However, if one wishes (a) to write the matching conditions in the equivalent Lichnerowicz form, which differs in that ‘admissible coordinates’, smooth across the boundary and always existent if the Darmois conditions are true and the spacetime is smooth enough (see Bonnor and Vickers (1981)), must be used, and (b) to tie the coordinates in the inner region to physical characteristics there, then some or all of the inessential parameters in the exterior may also be fixed. For vacua exterior to rotating shells or fluids, such matchings have been considered by, e.g., Bonnor (1980), Stachel (1984), Da Silva *et al* (1995a), Da Silva *et al* (1995b), Bonnor *et al* (1997).

These calculations prompt a criticism of the definition of cylindrical symmetry given in Kramer *et al* (1980), which is too restrictive.

2 The general form of the criterion

Proposition. *If the metric components f , m and ℓ in the Lewis metric form (1.4) are linearly dependent over \mathbb{R} , then by a homogeneous linear transformation of the*

coordinates (ϕ, X) with constant coefficients,
(i) the metric can be put in a locally diagonal form,

$$ds^2 = e^{2V}(dr^2 - \zeta dY^2) + \ell d\hat{\phi}^2 + \zeta f d\hat{X}^2, \quad (2.1)$$

where V , ℓ and f are functions of r and Y only, and it is manifest that there are two hypersurface-orthogonal Killing vectors, or
(ii) the metric has a null Killing vector and can locally be put in the form

$$ds^2 = e^{2V}(dr^2 + dz^2) + \ell d\hat{\phi}^2 + 2m d\hat{\phi} d\hat{t}, \quad (2.2)$$

where V , ℓ and m are functions of r and z only, or
(iii) the metric can locally be put in the form

$$ds^2 = e^{2V}(dr^2 + dz^2) + \rho[(\cos \psi d\hat{\phi} + \sin \psi d\hat{X})^2 - (-\sin \psi d\hat{\phi} + \cos \psi d\hat{X})^2] \quad (2.3)$$

where V , ρ and ψ are functions of r and z related to a Lewis form (1.4) with $\zeta = -1$ by $2V = \mu = \nu$, $\rho^2 = f\ell + m^2$, $\ell = f = \rho \cos(2\psi)$, $m = \rho \sin(2\psi)$. This last case has a pair of complex conjugate hypersurface-orthogonal Killing vectors.

Remark. The first of these possibilities includes the well-known static vacua associated with the names of Weyl, Kasner and Levi-Civita, and Lewis's first solution: note that if $\zeta = -1$ one of the KVs $\partial/\partial\phi$ and $\partial/\partial X$ in equation (2.1) must be timelike. The second possibility includes the limiting case of Lewis's solutions given explicitly by van Stockum though not in the form (2.2) (see equations (11.1) of van Stockum (1937), and the examples below), and the third includes Lewis's second solution (a complex continuation of the first) and can be seen to be a generalized form of the vacuum 'windmill' solutions discussed by McIntosh (1992).

Proof. To prove the proposition, I first consider those cases where the quadratic (1.7) is in some way degenerate. The linear dependence may arise simply because one of the three functions vanishes. If $m = 0$, clearly the metric is already in the form required for case (i). If $f = 0$, then $\partial/\partial X$ is a null KV, $\zeta = -1$ and $m \neq 0$ in order to have a non-degenerate metric of the correct signature, and the metric can be written in the form (2.2); if in addition ℓ/m is constant, ℓ can be set to zero by a further constant linear transformation of ϕ and t . If $\ell = 0$ then $\partial/\partial\phi$ is a null KV, and the form (2.2) can be obtained by exchanging ϕ and $X = t$.

Next I consider the case where only two of the functions appear with non-zero coefficients in the linear dependence. If the linear dependence takes the form $m = c\ell$ with $c \neq 0$, then $\hat{\phi} = \phi + cX$ diagonalizes the metric (note that ζf cannot be $c^2\ell$ or the metric is degenerate), and similarly if $m = cf$ with $c \neq 0$, $\hat{X} = X + c\phi$ diagonalizes the metric. The case where $f = c\ell$ is included in the general discussion of equation (1.6), as the special case where $a = 0$: however, dealing with it separately may help to clarify what happens in the more general case.

If $f = c\ell$, then by scaling X one can always set c to $\text{sgn}(c)$. If $\zeta = 1$, then f , ℓ and $f\ell - m^2$ must be positive in order for the metric to have correct signature, so $c > 0$ and one can take $c = 1$. The metric in the (ϕ, X) plane is then

$$\ell(d\phi^2 + dX^2) + 2m d\phi dX$$

and taking $\hat{X} = X + \phi$, $\hat{\phi} = X - \phi$ (agreeing with the roots ± 1 given by the quadratic (1.7)) gives the diagonal form

$$\frac{1}{2}[(m + \ell)d\hat{X}^2 + (\ell - m)d\hat{\phi}^2]$$

in which both metric coefficients must be positive. If $\zeta = -1$ and $c < 0$, one can take $c = -1$ and diagonalize the resulting metric in a similar way, except that now $m^2 > f\ell$ so one of the final metric coefficients is negative. If $\zeta = -1$ and $c > 0$, one can take $c = 1$, and then by defining ρ and ψ as in (iii) above, one can arrive at the form (2.3). In this case the quadratic (1.7) has roots $\pm i$, and the metric can be regarded as having complex conjugate HSO KVs, taking a diagonal form (involving the exponentials of $\pm 2i\psi$) with respect to the complex coordinates $\phi \pm iX$.

In the general case, where all three of f , ℓ and m enter the linear dependency relation with non-zero coefficients, one can make the coefficient of f equal to 1, and then inspect the resulting quadratic (1.7). If this has distinct real roots, case (i) applies and the coordinate transformation

$$X = \hat{X} + \hat{\phi}, \quad \phi = B\hat{X} + C\hat{\phi} \quad (2.4)$$

gives the metric a diagonal form, with $C^2\ell + 2mC + \zeta f$ and $B^2\ell + 2Bm + \zeta f$ as the new values of ℓ and ζf . The perhaps surprising form of the first part of equation (2.4) merely reflects the choice of scaling of $\hat{\phi}$: the form

$$X = \hat{X} + \hat{\phi}/C, \quad \phi = B\hat{X} + \hat{\phi}. \quad (2.5)$$

might be considered more natural. If the quadratic (1.7) has two equal real roots then $\partial_X + B\partial_\phi$ is a null Killing vector, and the coordinate transformation $\phi = \hat{\phi} + B\hat{X}$, $X = \hat{X}$ brings the metric into the form (2.2). Finally if the quadratic (1.7) has two complex conjugate roots the transformation (2.4) gives a complex diagonal form in which the new ℓ and ζf are complex conjugates. Writing these as $-\rho e^{2i\psi}$ and $-\rho e^{-2i\psi}$ respectively, setting $\sqrt{2}\hat{\phi} = u + iv$, $\sqrt{2}\hat{X} = u - iv$, and following this by renaming u as \hat{X} and v as $\hat{\phi}$, gives the form (2.3). Note that the form of the metric is much more easily established if one goes via the complex form than if one makes the corresponding real transformation directly. However, the overall coordinate transformation is strictly real: it is just

$$X = \sqrt{2}\hat{X}, \quad \phi = \sqrt{2}(b_1\hat{X} - b_2\hat{\phi}) \quad (2.6)$$

where b_1 and b_2 are respectively the real and imaginary parts of $B = C^*$. This completes the proof.

The forms (2.1) and (2.2) are unaffected by re-scaling each of the coordinates ϕ and X by constant factors, so at most two of the parameters of the allowed homogeneous linear transformations of these coordinates have been used. To preserve the form (2.3) the corresponding scalings have to match: this extra requirement could be removed by suitably putting an additional constant into the form (2.3). However, one would normally use the scalings to remove as far as possible any inessential parameters in the metrics (2.1)–(2.3), and in later sections we shall assume that such a scaling has been chosen in order to standardize the form of the locally equivalent metric considered: this is consistent with the form (2.3).

Thus when the functions f , ℓ and m are linearly dependent, the problem of solving the field equations can be simplified by solving the equations for the metrics (2.1)–(2.3)

above and then transforming. I have not attempted to characterize all cases where this applies. It does follow from the ansatz used in van Stockum (1937), section 9, which is that $v_{,D} = \Theta_{,D}(u + v^2)$, $u_{,D} = \Phi_{,D}(u + v^2)$ where $v = m/\ell$, $u = f/\ell$ and x^D is either r or Y (the field equations for vacuum give only that

$$\sum_D [ru_{,D}/(u + v^2)]_{,D} = 0 = \sum_D [rv_{,D}/(u + v^2)]_{,D},$$

which is consistent with but does not imply the ansatz in general). This ansatz has the consequence $v = Mu + N$ for constants M and N (or u or v may be zero or infinity, which van Stockum does not mention). For stationary cylindrically symmetric metrics of the form (1.2) whose energy-momentum components in the (ϕ, t) plane are a multiple of the metric in that plane this is not an ansatz but can be proved (see, e.g., Santos (1993)).

It may also be worth noting that in the form (1.3), $\rho = 0$ is usually referred to as the axis, though since the geometry may be singular there, the points may not be a properly-defined part of the Riemannian manifold (cf. Mars and Senovilla (1995)). If there is such an axis, if f , m and ℓ are linearly dependent, and if at the axis $f \neq 0$, then either $m = 0$ (and the metric was diagonal from the start), or the limiting value of f/m as $\rho \rightarrow 0$ necessarily gives one of the roots of the (possibly degenerate) quadratic (1.7).

To illustrate the result I look again at the examples studied by Castejon-Amenedo and MacCallum (1990). Three of these are in fact forms of Lewis's stationary cylindrically symmetric vacuum solutions. The Arbex and Som (1973) solutions are explicitly derived by a transformation from a static form (giving the 'Weyl class' treated in Da Silva *et al* (1995b), so called because they are related by the a transformation (2.5) to solutions in Weyl's class of static axisymmetric spacetimes), and are thus locally static, i.e. fall into case (i) above. (Whether these should be considered globally static, as claimed in Castejon-Amenedo and MacCallum (1990), is discussed below.) These solutions have

$$\begin{aligned} f &= \gamma^2(\exp(2\alpha) - \omega^2 r^2 \exp(-2\alpha)) \\ m &= \gamma^2 \omega (\exp(2\alpha) - r^2 \exp(-2\alpha)) \\ \ell &= \gamma^2(r^2 \exp(-2\alpha) - \omega^2 \exp(2\alpha)) \end{aligned} \quad (2.7)$$

where $\gamma^2(1 - \omega^2) = 1$, ω is constant, and α depends on r ; from this one can immediately see that $-f + (\omega + 1/\omega)m + \ell = 0$, the roots of the quadratic (1.7) are $\omega^{\pm 1}$, and the HSO KVs given in Castejon-Amenedo and MacCallum (1990) (which invert the construction of Arbex and Som (1973)) are recovered.

Bonnor's discussion (1980) of the van Stockum solutions (1937) shows that only in van Stockum's first case are the exterior solutions locally static. In this case, the functions in (1.4) in the coordinates used by Bonnor are

$$\begin{aligned} f &= (r \sinh(\epsilon - \theta) \operatorname{cosech} \epsilon)/R \\ m &= r \sinh(\epsilon + \theta) \operatorname{cosech} 2\epsilon \\ \ell &= (rR \sinh(3\epsilon + \theta) \operatorname{cosech} 2\epsilon \operatorname{sech} \epsilon)/2 \end{aligned} \quad (2.8)$$

where $\theta = \sqrt{1 - 4a^2 R^2} \log(r/R)$, $\tanh \epsilon = \sqrt{1 - 4a^2 R^2}$ and a and R are constants. It is obvious that f , m and ℓ depend linearly on the two functions $r \sinh \theta$ and $r \cosh \theta$. The dependence is

$$-Rf - 4 \cosh \epsilon \cosh(2\epsilon)m + 4 \cosh^2 \epsilon \ell/R = 0$$

with roots $-(2e^{\pm 2\epsilon} \cosh \epsilon)/R$ for the quadratic (1.7) and resulting HSO KVs as in Bonnor (1980), Castejon-Amenedo and MacCallum (1990).

In van Stockum's second case, the one not given explicitly by Lewis,

$$\begin{aligned} f &= r/R(1 - \log(r/R)) \\ m &= r(1 + \log(r/R))/2 \\ \ell &= rR(3 + \log(r/R))/4 \end{aligned} \quad (2.9)$$

and $-Rf - 4m + 4\ell/R = 0$, giving coincident roots $-2/R$ for the quadratic (1.7) and thus a null KV (as found by Bonnor (1980), Castejon-Amenedo and MacCallum (1990)); this is case (ii) above.

Finally, the third case is the same as the first but with all the hyperbolic functions replaced by the corresponding trigonometric ones, $\theta = \sqrt{4a^2R^2 - 1} \log(r/R)$ and $\tan \epsilon = \sqrt{4a^2R^2 - 1}$ ($0 \leq \epsilon < \pi/2$): this is equivalent to the complexified form given by Lewis. The linear dependency is $-Rf - 4 \cos \epsilon \cos(2\epsilon)m + 4 \cos^2 \epsilon \ell/R = 0$, the roots of the quadratic (1.7) are $-(2e^{\pm 2i\epsilon} \cos \epsilon)/R$ and case (iii) holds.

The paper of Som *et al* (1976) also considered the problem of stationary cylindrically symmetric vacuum metrics and derived the linear dependency directly (without first obtaining explicit forms for f , m and ℓ) in the manner mentioned above. They then concluded that all such metrics can be diagonalized, but this is only true if one allows complex coordinates as in case (iii), and overlooks the case (ii). In the notation of Som *et al* (1976), $m = \delta f + \gamma \ell$ and cases (ii) and (iii) arise when their $\eta = \sqrt{1 + 4\delta\gamma}$ becomes zero or negative.

Lastly, the metrics given by Piran *et al* (1986) have $\zeta = 1$,

$$f = \frac{[\alpha^2(1 - \lambda_u \lambda_v)^2 + (\lambda_u^2 + \lambda_v^2)]}{[(\alpha^2 \Xi^2 + (\lambda_u - \lambda_v)^2)]} \quad (2.10)$$

$$m = -\frac{a\sqrt{\alpha^2 - 1}[\Xi(\lambda_v + \lambda_u)^2]}{\sqrt{\lambda_u \lambda_v}[(\alpha^2 \Xi^2 + (\lambda_u - \lambda_v)^2)]} \quad (2.11)$$

$$\ell = (\rho^2 + m^2)/f \quad (2.12)$$

where a and $\alpha = M/a$ are constants, $\lambda_u = [\sqrt{a^2 + u^2} - u]/a$, $\lambda_v = [\sqrt{a^2 + v^2} + v]/a$, $u = t - \rho$, $v = t + \rho$, $\Xi = 1 + \lambda_u \lambda_v + 2[(1 - \alpha^{-2})\lambda_u \lambda_v]^{1/2}$ and a constant term in m/f has been removed for simplicity as this cannot affect whether (1.6) is satisfied, though as is shown in section 3 this is not an allowable transformation of an axisymmetric metric³. At $\rho = 0$, $u = v$, $\lambda_u \lambda_v = 1$, $\lambda_u + \lambda_v = 2\sqrt{(a^2 + t^2)}/a$, Ξ is a constant which is non-zero if $\alpha \neq 0$, and $\lambda_u^2 + \lambda_v^2 = 2(a^2 + 2t^2)/a^2$, so the denominators of f and m are equal and non-zero, and the numerators are constant multiples of $a^2 + 2t^2$ and $a^2 + t^2$ respectively; thus the ratio f/m at $\rho = 0$ cannot be constant as the criterion would demand (clearly in this case $m \neq 0$). Hence f , m and ℓ for this metric cannot be linearly dependent. This is not surprising since the metric is obtained by complex transformations from the Kerr metric, for which f , m and ℓ are linearly independent over \mathbb{R} (e.g. take the formulae for them as functions of the Boyer-Lindquist coordinates r and θ as in equation (19.19) of Kramer *et al* (1980)).

3 Axisymmetry and global restrictions

For a metric satisfying the conditions of the Proposition in the previous section, the transformations used to bring it into its canonical form work without difficulties when

³ Equation (67) of Castejon-Amenedo and MacCallum (1990) contains a typographical error in λ_v , but (68) is correct.

the (ϕ, X) coordinates can be considered to stretch to infinity in all directions. However, as the notation itself suggests, one is often interested in axisymmetric solutions in which the range of ϕ is restricted to $[0, 2\pi]$, and the (ϕ, X) surfaces thus have cylindrical topology. I now consider the restrictions this imposes on the permissible transformations.

First I consider the interpretation, in terms of the topological identification, of the parameters of the transformations to the local standard forms found in section 2. Let us consider transformations, similar to equation (2.5), of the form

$$X = \hat{X} + \hat{\phi}/S, \quad \phi' = Q\hat{X} + \hat{\phi}. \quad (3.1)$$

starting from coordinates $(\hat{\phi}, \hat{X})$, for the moment assumed to have infinite ranges, in which the metric takes the form (2.1). (The following argument requires adaptation for the other two cases but it is clear that the main arguments about the parameters needed to define a topological identification making the (ϕ, X) plane into a cylinder hold in all cases. Note, however, that if one starts from the form (2.2), exchange of $\hat{\phi}$ and \hat{X} may be required if the periodic coordinate is null.) Suppose that in these coordinates there is a topological identification, making the (ϕ', X) plane into a cylinder, in which ϕ' is periodic with period P' .

The identification of (ϕ', X) with $(\phi' + nP', X)$, for all integers n , identifies $(Q\hat{X} + \hat{\phi}, \hat{X} + \hat{\phi}/S)$ with $(Q\hat{X} + \hat{\phi} + nP', \hat{X} + \hat{\phi}/S)$. Then the origin is identified with $\hat{X} = -\hat{\phi}/S = nP'/(S - Q)$ and the lines $Q\hat{X} + \hat{\phi} = 0$ and $Q\hat{X} + \hat{\phi} = -nP'$ are identified. Thus two parameters (S and $P = SP'/(S - Q)$ say) can be regarded as defining a point to be identified with the origin, and the third (Q) as defining the lines along which the identification is made.

We can in addition to (3.1) make a re-scaling of ϕ' to make its period 2π , by $\phi = 2\pi\phi'/P'$, so the overall transformation is

$$X = \hat{X} + \hat{\phi}/S, \quad \phi = 2\pi(Q\hat{X} + \hat{\phi})/P'. \quad (3.2)$$

We can relate this to the form (2.5) by rescaling $\hat{\phi}$ by $2\pi/P'$ and setting $C = 2\pi S/P'$, $B = 2\pi Q/P'$.

From this description we can see that the parameter Q can be taken to be zero, since a point $(\hat{\phi}, \hat{X})$ is identified with all points $(\hat{\phi} + kP, \hat{X} - kP/S)$ where k is an integer, and we can in particular describe this by using the lines $\hat{\phi} = 0$ and $\hat{\phi} = kP$, and regarding S as fixing the amount by which points slip along those lines in the identification. The corresponding coordinate transformation is

$$X = \hat{X} + \hat{\phi}/S, \quad \phi = 2\pi\hat{\phi}/P. \quad (3.3)$$

Thus if, as discussed above, the coordinates in the standard forms have been scaled to remove as many inessential parameters as possible, all distinct axisymmetric geometries locally equivalent to the standard form can be obtained by applying a transformation (3.3) to the standard form (cf. Stachel (1982)) and considering ϕ to have period 2π . The number of essential parameters in the axisymmetric form is thus two more than the number in the corresponding locally equivalent standard form. The same argument applies, *mutatis mutandis*, if the standard form is itself taken to be axisymmetric: the standard form will then contain an essential parameter, \hat{P} say, allowing the period of $\hat{\phi}$ to be 2π , and distinct stationary forms will be given by different values of \hat{P}/P .

Correspondingly, given an axisymmetric metric in our class, we expect that transformations of the complementary form

$$\bar{X} = AX, \quad \bar{\phi} = \phi + HX \quad (3.4)$$

are allowed and are inessential in that they do not alter either the local geometry or the topological identification. As this statement disagrees with some earlier papers, the admissibility and properties of a transformation (3.4) will now be discussed in some detail. I first note that it alters the surfaces on which ϕ is constant, thus for example altering the X axis, but it preserves the surfaces on which X is constant, including the ϕ axis. One way to see directly that this transformation does not affect the geometry is to note that it leaves the vector $\partial/\partial\phi$ invariant, i.e. preserves the uniquely-defined Killing vector tangent to the closed curves, which is not preserved, in general, by (3.3). To clarify the point further I consider allowable coordinate changes directly.

Considering the angular coordinates ϕ and $\bar{\phi}$ to have a range 2π , the map (3.4) from ϕ to $\bar{\phi}$ is discontinuous at $\phi = 0$ (or 2π) and similarly at $\bar{\phi} = 0$. The transformation (3.4) is therefore sometimes regarded as inadmissible (see e.g. Da Silva *et al* (1995b)) or as destroying the periodicity of ϕ . However, periodic coordinates do not satisfy the requirement that a coordinate chart in an n -dimensional differential manifold should give a one-to-one map between an open set of the manifold and an open set in \mathbb{R}^n , and in my view the discontinuities therefore merely reflect the fact that identification of 0 and 2π makes the coordinates improper at those points. To construct a rigorous argument we should start by taking two or more true coordinate patches; transformations will then be allowable if after making them separately on each patch we can use the result to construct a new (improper) system of coordinates (ϕ, X) with a periodic ϕ .

We note that for this to be possible the curves on which \bar{X} is constant must be the closed curves on which X is constant, so, ignoring changes of origin, we must take \bar{X} proportional to X , i.e. among homogeneous linear transformations in the (ϕ, X) surfaces the first equation in (3.4) is the most general change in X consistent with periodicity of ϕ . Note that although I agree with other authors (e.g. Stachel (1982), Da Silva *et al* (1995b)) that a transformation $\bar{X} = AX + G\phi$ with $G \neq 0$ is disallowed, I do so on the grounds of the global inadmissibility of transforming X in such a way as to alter the identification implied, rather than the argument that the new X would have a periodic nature. This latter formulation does not take into account the fact that the usual ϕ , with 0 and 2π identified, is not strictly an admissible coordinate, i.e. the periodic X can only be derived by using improper coordinates in the first place (a periodic time that really was forced would of course be undesirable, except perhaps in situations with closed timelike lines).

The transformation $\bar{\phi} = \phi + HX$ only redefines ϕ differently on each curve on which X is constant, so one can see intuitively why it is allowed by envisaging taking a stack of rings and rotating each one by a different amount, which does not affect the actual geometry at all. An alternative way to describe the transformation (3.4) is that for both ϕ and $\bar{\phi}$ the range is infinite but ϕ and $\phi + 2n\pi$ are identified for any integer n (and similarly for $\bar{\phi}$).

In fact, if two or more true coordinate patches had been used to cover the manifold both before and after the transformation, each locally of the canonical form, the transformations on the overlaps between new and old coordinate regions would be smooth. For example, take patches U_1 and U_2 defined respectively by $-3\pi/4 < \phi < 3\pi/4$ and $\pi/4 < \phi < 7\pi/4$ and with ϕ coordinates denoted by ϕ_1 and ϕ_2 . The overlap $U_1 \cap U_2$

has two disjoint parts, on one of which $\pi/4 < \phi_1 = \phi_2 < 3\pi/4$ and on the other $5\pi/4 < \phi_2 = \phi_1 + 2\pi < 7\pi/4$. Now take a transformation (3.4) and use similar charts V_1 and V_2 defined by $\bar{\phi}$. The intersections, e.g. $V_1 \cap U_1$, each consist of a countable infinity of mutually disjoint pieces which can be labelled by the integer n required so that the corresponding transformation $\bar{\phi} = \phi + HX - 2\pi n$ gives values of $\bar{\phi}$ in the appropriate range, e.g. $(-3\pi/4, 3\pi/4)$ for V_1 , and this transformation is smooth on each piece. Thus this is an acceptable coordinate change.

One can of course apply a general linear transformation with constant coefficients to the ignorable coordinates in a true coordinate patch, but this would not respect the importance of the global topology. For example, the locally HSO KVs found in case (i) are not satisfactory as globally HSO KVs unless $m = 0$. If $m \neq 0$, the surfaces $\hat{\phi} = \text{constant}$ to which $\partial/\partial\hat{X}$ is orthogonal wind round the axis in a helix, and the integral curves of the Killing vector $\partial/\partial\hat{X}$ would meet such a surface infinitely often. (Note that this last means that the surfaces $\hat{X} = \text{constant}$ would not be achronal, despite being spacelike everywhere.) The term ‘static’ would normally be reserved for the case where the surfaces pass through, and are orthogonal to, the axis and meet each trajectory of $\partial/\partial X$ only once.

If (1.6) is satisfied with real B and C , then by a transformation of the form (3.4) with $H = -B$, we would have $\bar{\ell} = \ell$ and $\bar{m} = m + B\ell$, and the linear dependence would become $\bar{f} = (B - C)\bar{m}/\zeta$. This could then be related by a transformation (3.3) to a standard form (2.1). Similar remarks apply to cases (ii) and (iii).

The parameters in (3.4) can become essential if an additional identification is made on the X axis, i.e. in the toroidal case. Such possibilities are increased if there is a third Killing vector commuting with the first two, as in stationary cylindrically metrics (where one can take Y to be ignorable): then one can introduce further complications by making identification(s) in one or more variable(s) which identify the origin with a point in the three-dimensional (r, X, Y) space, and so on (cf. Tod (1994)).

4 Physical interpretation of the topological parameters

The parameters S and P in the transformation (3.3) do not change the Riemannian curvature tensor and its covariant derivatives at a given point, since locally they merely specify coordinate transformations (though they will of course alter the coordinate components of the tensors in the usual way). Hence they cannot affect the values of the Cartan scalars obtained in the procedure for local characterization of solutions of the Einstein equations (MacCallum and Skea, 1994; Paiva *et al.*, 1993; Da Silva *et al.*, 1995b), though they will in general alter the expressions for them in terms of the coordinates. However, they are invariant topological characteristics of the axisymmetric metric form, and it would be of interest to find a relation to curvature. As Stachel noted (1982), any such relation must come from the global holonomy⁴ of the solution, by taking closed curves around the axis, whose existence of course depends on the identification and of which the simplest, and the only ones which are trajectories of a Killing vector, are the curves on which r , X , and Y are constant and, using (3.3), ϕ runs from 0 to 2π . In general the equations for parallel transport of a vector \mathbf{v} along

⁴Unless otherwise stated, I consider only linear holonomy.

these curves give a set of four coupled linear homogeneous differential equations in the components of \mathbf{v} , first-order with respect to ϕ and with coefficients independent of ϕ .

For definiteness, take the case $\zeta = -1$ in the Lewis form (with $\mu = \nu = 2V$). The parallel transport equations to consider then take the form

$$\frac{d\mathbf{v}}{d\phi} = \mathbf{A}\mathbf{v} \quad (4.1)$$

where, if the components are given in terms of the coordinate numbering $(x^1, x^2, x^3, x^4) = (r, z, \phi, t)$, the only non-zero components of the matrix \mathbf{A} (i.e. the relevant Christoffel symbols $A^i_j = \{\Gamma^i_{j3}\}$) are

$$\begin{aligned} A^1_3 &= -\ell_{,r}/2e^{2V}, & A^1_4 &= -m_{,r}/2e^{2V}, \\ A^2_3 &= -\ell_{,z}/2e^{2V}, & A^2_4 &= -m_{,z}/2e^{2V}, \\ A^3_1 &= (f\ell_{,r} + mm_{,r})/2\rho^2, & A^3_2 &= (f\ell_{,z} + mm_{,z})/2\rho^2, \\ A^4_1 &= (m\ell_{,r} - \ell m_{,r})/2\rho^2, & A^4_2 &= (m\ell_{,z} - \ell m_{,z})/2\rho^2. \end{aligned} \quad (4.2)$$

In general the matrix \mathbf{A} will have four distinct eigenvectors \mathbf{v}_A with corresponding eigenvalues λ_A , and the general solution of equation (4.1) will be of the form

$$\mathbf{v}(\phi) = \sum_{A=1}^4 K_A \mathbf{v}_A \exp(\lambda_A \phi)$$

where the K_A are arbitrary functions of r and z : note that the λ_A and \mathbf{v}_A , although independent of ϕ (and t), will also in general depend on r and z . The objective is to examine how the parameters S and P affect the net change $\mathbf{v}(2\pi) - \mathbf{v}(0)$ round a circle.

The eigenvalue equation for \mathbf{A} takes the form

$$\lambda^4 + b_2\lambda^2 + b_4 = 0$$

where

$$\begin{aligned} b_4 &= \det \mathbf{A} = -(m_{,r}\ell_{,z} - m_{,z}\ell_{,r})^2/16e^{4V}\rho^2, \\ b_2 &= (g^{ab}\mathbf{w}_a^1\mathbf{w}_b^1 + g^{ab}\mathbf{w}_a^2\mathbf{w}_b^2)/4e^{2V}, \end{aligned}$$

and the one-forms \mathbf{w}^1 and \mathbf{w}^2 lie in the (ϕ, t) plane and have components $(\ell_{,r}, m_{,r})$ and $(\ell_{,z}, m_{,z})$ respectively in that plane. Thus the eigenvalues occur in pairs $\pm\sqrt{y}$ for y satisfying $y^2 + b_2y + b_4 = 0$; non-trivial holonomy arises unless for each λ_A , $\exp(2\lambda_A\pi) = 1$, and this would imply that each $\lambda_A = in_A$ for some integer n_A , so in particular b_4 would be positive.

If the metric is in case (i) with $\zeta = -1$, the transformation (3.3) is made, with P assumed to be positive, and F and L are the f and ℓ of the diagonal form, then $f = F$, $m = PF/2\pi S$, and $\ell = P^2(S^2L - F)/4\pi^2S^2$. Now the holonomy of the transformed metric is given by

$$b_2 = P^2[F(L_{,r}^2 + L_{,z}^2) - L(F_{,r}^2 + F_{,z}^2)/S^2]/(16\pi^2e^{2V}FL)$$

and

$$b_4 = -P^4(F_{,r}L_{,z} - F_{,z}L_{,r})^2/256S^2\pi^4e^{4V}FL.$$

It is easily seen that for given F and L , the values of P and S in general affect the holonomy. All the eigenvalues scale with P , as one would expect since this sets the

scale of ϕ relative to $\hat{\phi}$. S (or, more precisely, since a transformation $\bar{X} = -X$ alters nothing essential in the holonomy, $|S|$), also alters the set of eigenvalues: in particular two of the eigenvalues are zero when $S = \infty$. When $S \neq \infty$, $b_4 < 0$ in general, and in that case at least two of the eigenvalues are real and hence cannot lead to $\exp(2\lambda_A\pi) = 1$. (The two additional parameters of a general homogeneous linear transformation of coordinates, which appear in (3.4), cannot affect the holonomy since they do not affect the invariant definition of the curves or of parallel transport: however, as a check, an explicit computation was performed to confirm this.)

There are of course exceptional cases where the parameters do not affect the holonomy. If F and L are both constant, so that the spacetime is 2+2 decomposable, all the eigenvalues are zero and the holonomy is trivial. If the gradients of F and L are everywhere parallel so that F is a function of L (e.g. if both depend on only one of the coordinates r and z), at least two of the eigenvalues are zero. If in addition F is constant, S does not affect the holonomy; this arises in the case of flat space with identifications and in that case the linear holonomy depends only on P (the β of Tod (1994)), though S (his α or γ) does appear in the affine holonomy.

One can also see from the formulae above that, as one would expect, if $\hat{\phi}$ is scaled up by a factor K , P and C^2 scale up by the same factor and L by its inverse, while if \bar{X} is scaled up by a factor A , C^2 and F scale by its inverse.

To illustrate this, consider the case (2.8). There are a pair of zero eigenvalues, with one eigenvector in the z direction and another in a certain direction in the (ϕ, t) plane, and a pair of complex conjugate eigenvalues whose squares, obtained with the help of REDUCE, are

$$-R \sinh(5\epsilon + \theta) / (16re^{2V} \cosh^4 \epsilon \sinh \epsilon).$$

In these solutions $e^{2V} = (R/r)^{2a^2 R^2} \exp(-a^2 R^2)$. The values of ϵ and R (or a and R) are seen to affect the holonomy. One may expect that van Stockum's third case leads to similar formulae with trigonometric functions replacing the hyperbolic ones.

Although the additional parameters given by A and H in (3.4) are inessential, they may, as mentioned earlier, still be required if the solution is matched to an interior using Lichnerowicz's form of the matching conditions, see e.g. Bonnor *et al* (1997). The interior of course need not obey (1.6) even when the exterior does. If on the other hand, a metric covered by the earlier Proposition is continued to an axis, its essential parameters may be interpreted as properties of line masses. For example, Linet (1985) considered static cylindrically symmetric Einstein spaces and chose among them the ones interpretable as cosmic strings: thus only a line mass on the axis appeared.

In Da Silva *et al* (1995b), the parameters in the general vacuum solution of case (i) above, of which the metric (2.8) is a special case, are considered. The metric was taken in the form

$$\begin{aligned} f &= (a^2 n^2 - c^2 r^{2n}) / a n^2 r^{n-1} \\ m &= -(a^2 b n^2 + c(n - bc) r^{2n}) / a n^2 r^{n-1} \\ \ell &= (-a^2 b^2 n^2 + (n - bc)^2 r^{2n}) / a n^2 r^{n-1}, \end{aligned} \tag{4.3}$$

so $-f + ((n - 2bc)m + c\ell) / b(bc - n) = 0$, and the roots of the quadratic (1.7) are $-1/b$ and $-c/(bc - n)$, which is consistent with (3.7) and (3.8) of Da Silva *et al* (1995b). The root $B = -c/(bc - n) = -H$ can be used in the transformation (3.4), leading to a metric of the same form with $\bar{a} = n^2 a / (bc - n)^2$, $\bar{b} = b(bc - n)/n$, $\bar{c} = 0$ and $\bar{m} = \bar{b}\bar{f}$. The holonomy round the circles has two zero eigenvalues, as noted above, and the non-zero

eigenvalues are the square roots of

$$\frac{a^2 b^2 n^2 (n-1)^2 - (n+1)^2 (bc-n)^2 r^{2n}}{4an^2 e^V r^{n+1}}$$

where $e^{4V} = r^{n^2+1}$. The value of \bar{f} gives as our comparison metric, without any rescaling of t , the case with $F = n^2 a / (bc-n)^2 r^{n-1}$, $L = r^{n+1}$. Then $P^2 = 4\pi^2 (bc-n)^2 / n^2 a$ and $C^2 = a/b^2$. Scaling of \hat{X} to the perhaps more natural $F = 1/r^{n-1}$ would scale C^2 to $(bc-n)^2 / b^2 n^2$.

By matching to a shell, Stachel (1984) identified (using my notation) P with an energy density, n with stress-energy, and C with the rotation rate of the shell relative to a flat interior. Similarly, Da Silva *et al* (1995b) interpreted n as the Newtonian mass per unit length (or the total mass of a fluid interior region). In the case $c = 0$ and $n = 1$, they identified a and b as the energy density and angular momentum of a string on the axis (cf. Jensen and Kucera (1993)). By matching to a fluid interior they interpreted c as due to the vorticity. However, this arises because in their treatment the world-lines of the fluid flow in the interior invariantly define the t axis at the interface, and the use of admissible coordinates in matching, in the Lichnerowicz sense, then leads to the appearance of this quantity in form for the exterior space, although it is not an invariant of the exterior space.

Although the above discussion is based only on the case (2.1), similar results for holonomy will clearly hold in the other cases. Da Silva *et al* (1995a) correspondingly found similar results for parameter identification by matching for the case (iii) metric obtainable from the metric (4.3) by taking n to be pure imaginary.

5 Concluding remarks

The proposition in section 2 shows that linear dependence over \mathbb{R} of the metric components in canonical coordinates for a metric with an orthogonally-transitive commuting G_2 leads to (real or complex) HSO KVs, or a null KV, and hence to local coordinate transformations to one of the metric forms (2.1–2.3). These transformations contain in general four constant parameters (the coefficients in a linear transformation of the canonical coordinates with constant coefficients). In axisymmetric metrics, two of these parameters define the topological identification and can physically be identified from holonomy round circles along which only ϕ varies, and hence (assuming that ρ is single-valued, so that the concept of the interior is well-defined) with properties of the sources interior to a given value of ρ , including possible line sources on an axis.

This leaves the difficulty that, as Mars and Senovilla (1995) have pointed out, while one can develop a proper theory for regular axes (see Wilson and Clarke (1996) for example), such a theory for everywhere singular axes does not exist. In particular, one cannot always attach a well-defined meaning to statements such as ‘the field is axisymmetric about an infinite axis’ (Kramer *et al*, 1980). Taking $\ell > 0$ at some (r, Y) does not guarantee $\ell > 0$ for all (r, Y) but one usually wants to consider regions where $\ell > 0$ and Mars and Senovilla argue that one should require $\ell > 0$ in the neighbourhood of an axis. One may note, for example, that an initially diagonal form to which the transformation (3.3) has been applied may then have $\ell < 0$ near the axis even if $L = 0$ there. In practice fields which do not have regular axes are nevertheless routinely described as cylindrically symmetric, and in particular even those such solutions which

if continued to an axis would not be regular there may form part of a globally regular solution in which they are exteriors for some regular interior with different energy-momentum content: for example the solution (2.8) is an exterior for van Stockum's cylinders of rotating dust (van Stockum, 1937; Bonnor, 1980).

This possibility means that rather than associating the parameters of the topological identification with a line source on the axis, they may be associated with the physical characteristics of the source region to which the solution is matched, and if the matching is done in the Lichnerowicz form, it may also fix some inessential parameters of the exterior. It should be noted that the arguments of sections 3 and 4 can be extended, *mutatis mutandis*, to spacetimes not obeying the Proposition in Section 2.

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